PMM U.S.S.R.,Vol.44,pp.672-674 0021-8928/81/5 0672 $7.50/0

UDC 532.5

NOTE ON THE FORMULATION OF THE PROBLEM OF FLOW THROUGH A BOUNDED REGION USING EQUATIONS OF PERFECT FLUID*

A. V. KAZHIKHOV

The problem of perfect incompressible fluid flow through a bounded region of space is considered. It is shown that for obtaining an unambiguous local solution of Euler's equation it is sufficient to define along the whole boundary of the flow region the normal component of the velocity vector and supplement this at the inflow section by two tangent components of the vortex vector.

This note is a supplement to the paper /1/ in which it was suggested that the problem of perfect fluid flow through a bounded region of space should contain in its formulation besides the definition of the normal velocity vector the definition of three components of the vortex vector at the inflow section.

As in /1/, V is the simply connected bounded flow region with cylindrical boundary S consisting of three parts: the lateral surface S_0, the inflow section S_1, and the outflow section S_2. We denote by $\mathbf{x} = (x_1, x_2, x_3)$ the Cartesian coordinates of points of V, by t the time, $t \in [0, T], 0 < T < \infty$, by $\mathbf{v} = (v_1, v_2, v_3)$ the velocity vector, and by $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ the velocity vortex vector. We assume that $S_i\ (i = 0, 1, 2)$ are surfaces of class $C^{2+\alpha}\ (0 < \alpha < 1)$ and that S_0 joins S_1 and S_2 along curves L_1 and L_2 at right angle. Let us consider Euler's equation for a perfect fluid, expressed in terms of the vortex (see /2/)

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} = \mathbf{f}, \quad \operatorname{rot}\mathbf{v} = \boldsymbol{\omega}, \ \operatorname{div}\mathbf{v} = 0 \tag{1}$$

We assume that at the initial instant of time $t = 0$ the velocity field is known and that at the boundary S of region V is specified the normal velocity component

$$\mathbf{v} = \mathbf{v}_0(\mathbf{x}), \ t = 0, \ \mathbf{x} \in V, \ \operatorname{div}\mathbf{v}_0 = 0 \tag{2}$$

$$(\mathbf{v}\cdot\mathbf{n}) = 0 \text{ on } S_0, \quad (\mathbf{v}\cdot\mathbf{n}) = g_1 > 0 \text{ on } S_1, \ (\mathbf{v}\cdot\mathbf{n}) = g_2 < 0 \text{ on } S_2 \tag{3}$$

where $\mathbf{n}$ is the unit vector of the inward normal to S, and g_1 and g_2 are specified functions on S_1 and S_2, respectively.

Vector $\boldsymbol{\omega}$

$$\boldsymbol{\omega} = \mathbf{h}(\mathbf{x}, t), \ \mathbf{x} \in S_1$$

was specified in /1/ in addition to conditions (3) along section S_1.

Let us show that the specification of three components of vortex results in an overdetermined boundary value problem. We assume for simplicity that S_1 is a plane cross section normal to the x_1-axis, and take, as an example, the following boundary values on S_1:

$$v_1 = (\mathbf{v}\cdot\mathbf{n}) \equiv 1, \ \omega_1 = (\boldsymbol{\omega}\cdot\mathbf{n}) \equiv 0 \text{ on } S_1$$

Then from the first equation of system (1) we obtain

$$\frac{\partial \omega_1}{\partial x_1} = f_1 \text{ on } S_1$$

On the other hand, since $\boldsymbol{\omega} = \operatorname{rot}\mathbf{v}$, hence $\operatorname{div}\boldsymbol{\omega} = 0$, and consequently

$$\frac{\partial \omega_1}{\partial x_1} = -\frac{\partial h_2}{\partial x_2} - \frac{\partial h_3}{\partial x_3}$$

which shows that vector $\mathbf{h}$ cannot be arbitrary.

The overdetermination of the problem results in the inaccuracy of the proof of the theorem of existence in /1/, since the method of successive approximations does not ensure the fulfillment of the equality $\operatorname{div}\boldsymbol{\omega}^k = 0$ at every iteration step.

It appears that for a correct formulation of the problem it is sufficient to specify on S_1 only the tangent components of vector $\boldsymbol{\omega}$, i.e.

$$(\mathbf{w}\cdot\boldsymbol{\tau}_i) = h_i(x, t), \quad \mathbf{x} \in S_1, \quad i = 2, 3 \tag{4}$$

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*Prikl.Matem.Mekhan.,44,No.5,947-950,1980

where τ_i are linearly independent vectors tangent to S_1. To prove the existence of a solution we can apply the method of iteration used in /1/, taking as the zero approximation the input data and, when determining the k-th approximation solve the linear system of equations

$$\frac{\partial \boldsymbol{\omega}^k}{\partial t} + (\mathbf{v}^{k-1} \cdot \nabla)\, \boldsymbol{\omega}^k - (\boldsymbol{\omega}^k \cdot \nabla)\, \mathbf{v}^{k-1} = \mathbf{f} \tag{5}$$

$$\operatorname{rot} \mathbf{v}^k = \boldsymbol{\omega}^k, \ \operatorname{div} \mathbf{v}^k = 0 \tag{6}$$

with initial and boundary conditions (2)—(4). It is assumed that the specified functions have the following properties of smoothness

$$\mathbf{f} \in C^{1+\alpha}(Q), \quad Q = V \times (0, T), \ \operatorname{div} \mathbf{f} = 0, \ \mathbf{v}_0 \in C^{2+\alpha}(V), \ \ g_i \in C^{2+\alpha, 1+\alpha}(S_i^T), \ i = 1, 2, \ h_j \in C^{1+\alpha}(S_1^T), \ j = 2, 3$$

$$S_m^T = S_m \times (0, T), \ \ m = 0, 1, 2$$

satisfy the congruence condition, and the solution is understood in the classical sense, i.e.

$$\mathbf{v} \in C^{2+\alpha, 1+\alpha}(Q), \ \ \boldsymbol{\omega} \in C^{1+\alpha}(Q)$$

To determine vector $\boldsymbol{\omega}^k$ from the first order equations (5) it is necessary to know the boundary values on S_1, including that of the normal component $(\boldsymbol{\omega}^k \cdot \mathbf{n})$. For the determination of $\mathbf{v}^k$ with the use of $\boldsymbol{\omega}^k$ and conditions (3) it is necessary and sufficient that $\operatorname{div} \boldsymbol{\omega}^k = 0$. For determining the field of vortex $\boldsymbol{\omega}^k$ it is, thus, necessary to solve the overdetermined system consisting of Eqs. (5) and equation $\operatorname{div} \boldsymbol{\omega}^k = 0$.

The singularity of such system is that on any smooth hypersurface Γ in the space R^4 of variables $(\mathbf{x}, t)$ there is a relation between the derivatives of components of vector $\boldsymbol{\omega}^k$ taken along Γ. Let us introduce on Γ the local system of coordinates $\sigma_i = \sigma_i(\mathbf{x}, t), i = 0, 1, 2, 3,$ with the σ_0-axis normal to Γ. Then from Eqs. (5), after scalar multiplication by $\nabla\sigma_0$, we obtain

$$\left(\frac{d\boldsymbol{\omega}^k}{dt} \cdot \nabla\sigma_0\right) = ((\mathbf{f} + (\boldsymbol{\omega}^k \cdot \nabla)\, \mathbf{v}^{k-1}) \cdot \nabla\sigma_0), \quad \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v}^{k-1} \cdot \nabla)$$

We write this equality in new variables σ, using equation $\operatorname{div} \boldsymbol{\omega} = 0$ which in new coordinates is of the form

$$\left(\frac{\partial \boldsymbol{\omega}^k}{\partial \sigma_0} \cdot \nabla\sigma_0\right) = -\sum_{i=1}^{3} \left(\frac{\partial \boldsymbol{\omega}^k}{\partial \sigma_i} \cdot \nabla\sigma_i\right)$$

and obtain the relation

$$\sum_{i=1}^{3} \frac{\partial \boldsymbol{\omega}^k}{\partial \sigma_i} \left(\frac{d\sigma_i}{dt} \nabla\sigma_0 - \frac{d\sigma_0}{dt} \nabla\sigma_i\right) = ((\mathbf{f} + (\boldsymbol{\omega}^k \cdot \nabla)\, \mathbf{v}^{k-1}) \cdot \nabla\sigma_0) \tag{7}$$

which contains derivatives of $\boldsymbol{\omega}^k$ only on the tangents to Γ in directions σ_i $(i = 1, 2, 3)$. Thus, when two of the three components of vector $\boldsymbol{\omega}^k$ are specified on Γ, the third must be determined by Eq. (7). If Γ is the inflow section of S_1^T, it is possible to set $\sigma_i = \sigma_i(\mathbf{x})$ $(i = 0, 1, 2)$ and $\sigma_3 = t$. Then $\nabla\sigma_0 = \mathbf{n}$, and formula (7) assumes the form

$$\frac{\partial \omega_n^k}{\partial t} + \sum_{i=1}^{2} \frac{\partial}{\partial \sigma_i} (\omega_n^k v_{\sigma_i}^{k-1} - v_n^{k-1} \omega_{\sigma_i}^k) + \left((v_n^{k-1} \boldsymbol{\omega}^k - \omega_n^k \mathbf{v}^{k-1}) \cdot \sum_{i=1}^{2} \frac{\partial \nabla\sigma_i}{\partial \sigma_i}\right) + \sum_{i=1}^{2} \left((\omega_{\sigma_i}^k \mathbf{v}^{k-1} - v_{\sigma_i}^{k-1} \boldsymbol{\omega}^k) \cdot \frac{\partial \mathbf{n}}{\partial \sigma_i}\right) = f_n \tag{8}$$

The following notation is used here: if $\mathbf{u}$ is $\boldsymbol{\omega}^k, \mathbf{v}^{k-1}$ or $\mathbf{f}$, then

$$u_n = (\mathbf{u} \cdot \mathbf{n}), \ u_{\sigma_i} = (\mathbf{u} \cdot \nabla\sigma_i), \ i = 1, 2$$

Since the tangent components of vector $\boldsymbol{\omega}^k$ are known on S_1, Eq. (8) makes it possible to determine the normal component $\omega_n^k = h_1^k$. Note that the boundary L_1 of section S_1 is a characteristic of Eq. (8), and it is unnecessary to impose any further conditions on L_1, i.e. the solution is constructed on the basis of initial data only. Having obtained h_1^k, we can determine vector $\boldsymbol{\omega}^k$ in the whole flow region $Q = V \times (0, T)$, taking as the boundary conditions for the component $(\boldsymbol{\omega}^k \cdot \mathbf{n})$ on S_1 the obtained function h_1^k.

Such construction ensures the fulfillment of equality $\operatorname{div} \boldsymbol{\omega}^k = 0$ in Q. Indeed, applying the divergence operation to Eq. (5) for $\boldsymbol{\omega}^k$, we find that the scalar function $\operatorname{div} \boldsymbol{\omega}^k$ satisfies the homogeneous linear equation

$$\frac{d}{dt} (\operatorname{div} \boldsymbol{\omega}^k) = 0$$

since $\operatorname{div} \mathbf{f} = 0$. From initial data we have $\operatorname{div} \boldsymbol{\omega}^k = 0$ at $t = 0$ and from Eqs. (5) and (8) we have $\operatorname{div} \omega^k = 0$ at the boundary of S_1 where $(\omega^k . n) = h_1^k$, consequently, $\operatorname{div} \omega^k \equiv 0$ in Q. Equations (3) and (6) with the obtained ω^k unambiguously determine vector v^k.

Thus, when the tangent components of vortex ω are specified at the inflow section, the method of proving the existence of solution of problem (1)—(4) used in /1/ is supplemented by one more step consisting of the determination of boundary values of the vortex normal component at section S_1 using Eq.(8). All calculations related to the theoretical determination of estimates, including those for the solution of Eq.(8), follow those in /1/. Uniqueness of the solution of problem (1)—(4) is proved by the conventional method of constructing a homogeneous linear problem for the remainder of the two possible different solutions and applying the same estimates as in the theorem of existence.

In concluding, we would point out that the problems for Eqs.(1) may be formulated differently from (2)—(4). For example, one could specify on S_1 the normal components of vectors ω and $\operatorname{rot} \omega$. Specifying $(\operatorname{rot} \omega \cdot n)$ and using formula (8) we obtain an elliptic system of first order equations for the tangent components ω_{σ_i} $(i = 1, 2)$ and vortex ω. A boundary value problem of the Riemann—Hilbert type is obtained for ω_{σ_1} and ω_{σ_2} from the condition of merging of sections S_1 and S_0 along line L_1. Finally, when on S_1 the normal and one of the tangent components of vortex ω are known, then the second tangent component is obtained from Eq.(8). It is then necessary to specify on a part of curve L_1 supplementary conditions in conformity with the known rules for formulating boundary value problems for first order equations.

The author thanks O. A. Ladyzhenskaia and L. V. Ovsiannikov for discussions of this subject.

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Translated by J.J.D.